Renormalization group treatment of correlated percolation

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A renormalization group procedure is developed for correlated site percolation lattices that can be described by Gaussian statistics. The procedure is a generalization of the procedure for independent site percolation lattices. As an example, a simple renormalization group procedure is generalized from the independent triangular lattice to the correlated triangular lattice. The procedure gives the exact percolation threshold $p_c=\frac{1}{2}$ for both the independent and the correlated triangular lattice.

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I. INTRODUCTION

In site percolation, one chooses the sites of a lattice to be open or closed by means of a statistical algorithm. The site percolation problem is then to determine the dependence of statistical features of clusters of open sites on the type of lattice and the algorithm. A particularly important quantity is the fraction of open sites at which there is a connected cluster of open sites that spans the lattice. This is denoted the percolation probability p_c. The traditional percolation algorithm is to choose the sites independently to be open with probability p and closed with probability $1-p$. Investigation of the percolation statistics of independent site lattices has received a great deal of attention for many years, and many approaches have been developed [1,2]. Among these are renormalization group methods [3].

There is a general approach that allows for both independent and correlated site probabilities. One sets up a random function $I(\mathbf{r})$, defined at the position $\mathbf{r}$ of each lattice site, and chooses a threshold value I_T. The lattice site at position $\mathbf{r}$ is then taken to be open if $I(\mathbf{r})\leqslant I_T$, and to be closed if $I(\mathbf{r})>I_T$. Of course, one could equally well reverse the inequalities. The independent site lattice is obtained by choosing $I(\mathbf{r})$ at the lattice sites to be independent random variables drawn from identical probability distributions.

The purpose of this paper is to describe a real space renormalization group procedure that can be applied to correlated lattices characterized by functions $I(\mathbf{r})$ that are described by Gaussian statistics. Gaussian statistics do not describe the most general correlations between lattice sites. Nevertheless, Gaussian statistics describe a type of lattice which forms a useful generalization of independent lattices. Section II describes the functions $I(\mathbf{r})$ and their probability distributions. Section III develops the renormalization group procedure. The procedure is illustrated by using a simple renormalization group to determine the percolation threshold of the triangular lattice.

II. PROBABILITY DISTRIBUTIONS

Consider a periodic d-dimensional lattice having N sites along each edge for a total of $\mathfrak{N}=N^d$ sites. Label the sites along the ith edge ($i=1,\ldots,d$) by m_i, $0\leqslant m_i\leqslant N-1$, so that each lattice site is labeled by the set $\{m_i\}$. It will be convenient to relabel the lattice sites by the single index

$$m=\sum_{i=1}^{d} m_i N^{i-1}, \tag{1}$$

where $0\leqslant m\leqslant \mathfrak{N}-1$. Let the lattice sites be located at the positions $\mathbf{r}_m$, and let $I_m\equiv I(\mathbf{r}_m)$. The statistics of the I_m are completely characterized by the $\mathfrak{N}$ point density function [4] $P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})$ and the $\mathfrak{N}$ point distribution function

$$F_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1}) = \int_{-\infty}^{I_0}\cdots\int_{-\infty}^{I_{\mathfrak{N}-1}} P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})\,dI_0\cdots dI_{\mathfrak{N}-1}. \tag{2}$$

Here $P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})dI_0\cdots dI_{\mathfrak{N}-1}$ is the probability that the mth random variable lies between I_m and I_m+dI_m, $m=0,\ldots,\mathfrak{N}-1$, and $F_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})$ is the probability that the mth random variable is less than or equal to I_m.

For independent sites, the $\mathfrak{N}$ point density and distribution functions are products of one point functions $P_1(I_m)$ and $F_1(I_m)$. The most natural choice of the density function for a percolation lattice, and the choice generally made for simulations of percolation, is to take P_1 to be uniform on the interval $[0,1]$, i.e.,

$$P_1(I_m)=\begin{cases} 0, & I_m<0 \\ 1, & 0\leqslant I_m\leqslant 1 \\ 0, & I_m>1. \end{cases} \tag{3}$$

Then the site m is open with probability p and closed with probability $1-p$, where

$$p=F_1(I_T)=\begin{cases} 0, & I_T<0 \\ I_T, & 0\leqslant I_T\leqslant 1 \\ 1, & I_T>1. \end{cases} \tag{4}$$

However, it is not necessary to choose this density function. At least, any piecewise continuous density function $P_1(I_m)$ will give the site m open with probability $p=F_1(I_T)$ and closed with probability $1-p$.

It is most useful to choose the density function to be a Gaussian function with mean 0 and variance 1, i.e.,

$$P_1(I_m)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{I_m^2}{2}\right). \tag{5}$$

Then

$$p=F_1(I_T)=\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{I_T}{\sqrt{2}}\right)\right], \tag{6}$$

where $\mathrm{erf}(x)$ is the error function [5]:

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy. \tag{7}$$

Multiplying the one point probability densities for the lattice sites gives the $\mathfrak{N}$ point probability density

$$P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})=\frac{1}{(2\pi)^{\mathfrak{N}/2}}\exp\left(-\frac{1}{2}\sum_{m=0}^{\mathfrak{N}-1}I_m^2\right). \tag{8}$$

To generalize this treatment to correlated lattices, choose the I_m to have a general Gaussian distribution with $\mathfrak{N}$ point probability density

$$P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})=\frac{1}{(2\pi)^{\mathfrak{N}/2}|Q|^{1/2}}\exp\left(-\frac{1}{2}\sum_{m=0}^{\mathfrak{N}-1}\sum_{n=0}^{\mathfrak{N}-1}Q_{mn}^{-1}I_mI_n\right) \tag{9}$$

Here I_m all have mean 0, Q is the covariance matrix,

$$Q_{mn}=\langle I_mI_n\rangle, \tag{10}$$

which is required to be positive definite [6], and $|Q|$ is the determinant of Q. Since the discussion is limited to periodic lattices,

$$Q_{mn}\equiv Q(\mathbf{r}_m,\mathbf{r}_n)=Q(|\mathbf{r}_m-\mathbf{r}_n|). \tag{11}$$

This can be written

$$Q(|\mathbf{r}_m-\mathbf{r}_n|)=Q_0q(|\mathbf{r}_m-\mathbf{r}_n|), \tag{12}$$

where $Q_0=Q(0)$ and $|q(|\mathbf{r}_m-\mathbf{r}_n|)|\leq 1$ [7]. Finally we require that $q(|\mathbf{r}_m-\mathbf{r}_n|)\to 0$ as $|\mathbf{r}_m-\mathbf{r}_n|\to\infty$.

In working through the renormalozation group, it will be necessary to determine probability densities $P_{\mathfrak{K}}(I_0,\ldots,I_{\mathfrak{K}-1},)$, $\mathfrak{K}<\mathfrak{N}$. This is most conveniently done by using the characteristic function, the Fourier transform of the probability density [4],

$$M_{\mathfrak{N}}(v_0,\ldots,v_{\mathfrak{N}-1})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(i\sum_{m=0}^{\mathfrak{N}-1}I_mv_m\right)\times P_{\mathfrak{N}}(I_0,\ldots,I_{\mathfrak{N}-1})dI_0\cdots dI_{\mathfrak{N}-1}. \tag{13}$$

Since

$$M_{\mathfrak{K}}(v_0,\ldots,v_{\mathfrak{K}-1})=M_{\mathfrak{N}}(v_0,\ldots,v_{\mathfrak{K}-1},0,\ldots,0), \tag{14}$$

$P_{\mathfrak{K}}(I_0,\ldots,I_{\mathfrak{K}-1})$ is obtained by setting $v_m=0$, $m=\mathfrak{K},\ldots,\mathfrak{N}-1$ and taking the inverse Fourier transform. With Eq. (9), a straightforward calculation gives

$$M_{\mathfrak{N}}(v_0,\ldots,v_{\mathfrak{N}-1})=\exp\left(-\frac{1}{2}\sum_{m=0}^{\mathfrak{N}-1}\sum_{n=0}^{\mathfrak{N}-1}Q_{mn}v_mv_n\right). \tag{15}$$

III. RENORMALIZATION GROUP

A real space renormalizaton group procedure on a percolation lattice consists of three steps [3].

(1) The lattice is divided into identical blocks of sites.

(2) Each block of sites is replaced by a single site on a renormalized lattice with renormalized probabilities.

(3) The lattice spacing of the renormalized lattice is brought to the same value as the lattice spacing of the original lattice.

The overall effect of this procedure is to replace the original lattice by an identical lattice with renormalized probabilities. Since the original and renormalized lattices are identical, one can take the same $\mathfrak{N}$ point density function [Eq. (9)] for both. However, the threshold I_T will be renormalized.

The application of the renormalization procedure—that is, the way of choosing the lattice blocks and renormalized probabilities—depends on the type of lattice. As an example and test of the procedure for correlated lattices, consider the percolation threshold of the triangular lattice. The exact value for the percolation threshold of the triangular lattice can be deduced by an argument of Sykes and Essam [8], (also see Ref. [9], p. 211) [10]. They showed that for any plane lattice L there is a ''matching lattice'' L^*, such that

$$p_c(L)+p_c(L^*)=1. \tag{16}$$

The triangular lattice is self matching so Eq. (16) implies that $p_c=1/2$. Essam [9] (p. 211) pointed out that the argument is valid for correlated as well as independent lattices.

For independent triangular lattices there is a simple renormalization group procedure that leads to the exact result for the percolation threshold [3] (Sec. 3.4). I will show that the same procedure leads to the exact percolation threshold for correlated triangular lattices. In the simple procedure the original lattice is broken into triangular blocks of three sites each. The corresponding site on the renormalized lattice is open if a majority, i.e., two or three, of the sites on the block are open. Label the sites of the original block by the indices 0, 1, and 2, and the renormalized site by the index 0. Then the renormalization condition is

$$\Pr(I_0'\leq I_T')=3\Pr(I_0\leq I_T,I_1\leq I_T,I_2>I_T)+\Pr(I_0\leq I_T,I_1\leq I_T,I_2\leq I_T), \tag{17}$$

where $\Pr(I_0'\leq I_T')$ is the probability that $I_0'\leq I_T'$, etc. The first term on the right is the sum of the three identical probabilities that two of the original sites are open and one closed. The second term is the probability that all three of the original sites are open. Now

$$\Pr(I_0 \leq I_T, I_1 \leq I_T, I_2 > I_T) = \Pr(I_0 \leq I_T, I_1 \leq I_T) - \Pr(I_0 \leq I_T, I_1 \leq I_T, I_2 \leq I_T), \tag{18}$$

where the first term on the right is the probability that two sites are open regardless of the other site. Combining this with Eq. (17) gives

$$\Pr(I_0' \leq I_T') = 3\Pr(I_0 \leq I_T, I_1 \leq I_T) - 2\Pr(I_0 \leq I_T, I_1 \leq I_T, I_2 \leq I_T). \tag{19}$$

For an independent lattice, Eq. (19) becomes

$$p' = 3p^2 - 2p^3, \tag{20}$$

which has fixed points $p' = p$, at $p = 0$, 1, and 1/2. For a correlated lattice, Eq. (19) becomes, in terms of the distribution functions,

$$F_1(I_T') = 3F_2(I_T, I_T) - 2F_3(I_T, I_T, I_T). \tag{21}$$

This equation has fixed points $I_T' = I_T$, at $I_T = -\infty$, $p = F_1(-\infty) = 0$, and $I_T = \infty$, $p = F_1(\infty) = 1$. I will show that it also has a fixed point at $I_T = 0$: $p = F_1(0) = 1/2$.

To investigate Eq. (21), it is necessary to determine the corresponding probability densities which can be done from the characteristic functions. In Eq. (15), set all $v_m = 0$ except for v_0, v_1, and v_2. The only elements of the q matrix then remaining are $q(0) = 1$ and $q(a) = q$, where a is the lattice spacing. This gives

$$M_3(v_0, v_1, v_2) = \exp\left\{ -\frac{1}{2} Q_0 [(v_0^2 + v_1^2 + v_2^2) - 2q(v_0 v_1 + v_1 v_2 + v_2 v_0)] \right\}. \tag{22}$$

Obtain $M_2(v_0, v_1)$ from Eq. (22) by setting $v_2 = 0$, and $M_1(v_0)$ by also setting $v_1 = 0$. Then take the inverse Fourier transforms to obtain

$$P_1(I_0) = \frac{1}{\sqrt{2\pi Q_0}} \exp\left[-\frac{I_0^2}{2Q_0} \right], \tag{23}$$

$$P_2(I_0, I_1) = \frac{1}{2\pi Q_0 \sqrt{1-q^2}} \exp\left[-\frac{I_0^2 - 2qI_0 I_1 + I_1^2}{2Q_0(1-q^2)} \right], \tag{24}$$

and

$$P_3(I_0, I_1, I_2) = \frac{1}{(2\pi Q_0)^{3/2}(1-q)\sqrt{1+2q}} \exp\left[-\frac{(1+q)(I_0^2 + I_1^2 + I_2^2) - 2q(I_0 I_1 + I_1 I_2 + I_2 I_0)}{2Q_0(1-q)(1+2q)} \right]. \tag{25}$$

From Eq. (23),

$$F_1(0) = \int_{-\infty}^{0} P_1(I_0)\, dI_0 = \frac{1}{2}. \tag{26}$$

From Eq. (24),

$$F_2(0,0) = \int_{-\infty}^{0}\int_{-\infty}^{0} P_2(I_0, I_1)\, dI_0 dI_1 = \int_{0}^{\infty}\int_{0}^{\infty} P_2(I_0, I_1)\, dI_0 dI_1, \tag{27}$$

where the last equality is obtained by letting $I_m \to -I_m$, $m = 0$, and 1. On making the change of variables

$$I_0 = \sqrt{\frac{1+q}{2}}\, x - \sqrt{\frac{1-q}{2}}\, y, \tag{28}$$

$$I_1 = \sqrt{\frac{1+q}{2}}\, x + \sqrt{\frac{1-q}{2}}\, y$$

in $P_2(I_0, I_1)$, Eq. (27) becomes

$$F_2(0,0) = \frac{1}{2\pi Q_0} \int\int e^{-(1/2Q_0)(x^2+y^2)} dx\, dy, \tag{29}$$

where the region of integration is bounded by the lines obtained by setting $I_0 = 0$ and $I_1 = 0$ in Eq. (28). Changing variables to polar coordinates gives

$$F_2(0,0) = \frac{\Theta}{2\pi Q_0} \int_0^{\infty} e^{-(r^2/2Q_0)} r\, dr = \frac{\Theta}{2\pi}, \tag{30}$$

where Θ is the angle between the lines $I_0 = 0$ and $I_1 = 0$. On setting $I_0 = 0$ and $I_1 = 0$, Eq. (28) takes the form $\mathbf{n}_i \cdot \mathbf{r} = 0$, $i = 0$ and 1, where $\mathbf{n}_i$ is a unit normal to line i. If Φ is the angle between the two normals, $\Theta = \pi - \Phi$, and

$$\cos\Phi = -\cos\Theta = \sin\left(\Theta - \frac{\pi}{2} \right) = \mathbf{n}_0 \cdot \mathbf{n}_1 = q, \tag{31}$$

so $\Theta = (\pi/2) + \sin^{-1} q$ and

$$F_2(0,0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} q. \tag{32}$$

From Eq. (25),

$$F_3(0,0,0,) = \int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0} P_3(I_0, I_1, I_2)\, dI_0 dI_1 dI_2 = \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} P_3(I_0, I_1, I_2)\, dI_0 dI_1 dI_2, \tag{33}$$

where the last equality is obtained by letting $I_m \rightarrow -I_m$, $m=0$, 1, and 2. On making the change of variables

$$I_0 = \sqrt{\frac{1+2q}{3}}\,x + \sqrt{\frac{1-q}{6}}\,y + \sqrt{\frac{1-q}{2}}\,z,$$

$$I_1 = \sqrt{\frac{1+2q}{3}}\,x + \sqrt{\frac{1-q}{6}}\,y - \sqrt{\frac{1-q}{2}}\,z, \qquad (34)$$

$$I_2 = \sqrt{\frac{1+2q}{3}}\,x - 2\sqrt{\frac{1-q}{6}}\,y$$

in $P_3(I_0,I_1,I_2)$, Eq. (33) becomes

$$F_3(0,0,0) = \frac{1}{(2\pi Q_0)^{3/2}} \int\int\int \times e^{-(1/2Q_0)(x^2+y^2+z^2)}\,dx\,dy\,dz, \qquad (35)$$

where the region of integration is bounded by the planes obtained by setting $I_0=0$, $I_1=0$, and $I_2=0$ in Eq. (34). Changing variables to polar coordinates gives

$$F_3(0,0,0) = \frac{\Omega}{(2\pi Q_0)^{3/2}} \int_0^\infty e^{-r^2/2Q_0} r^2 dr = \frac{\Omega}{4\pi}, \qquad (36)$$

where Ω is the solid angle bounded by the planes $I_0=0$, $I_1=0$, and $I_2=0$. On setting $I_0=0$, $I_1=0$, and $I_2=0$, Eq. (34) takes the form $\mathbf{n}_i \cdot \mathbf{r}=0$, $i=0$, 1, and 2, where $\mathbf{n}_i$ is a unit normal to plane i.

The solid angle Ω is equal to the area of the spherical triangle formed by the intersections of the three planes with a unit sphere centered at the origin. A theorem of spherical geometry [11] gives this area to be the sum of the three angles of the triangle minus π. Since each angle of the triangle is just equal to the dihedral angle between the corresponding planes,

$$\Omega = \Theta_{01} + \Theta_{12} + \Theta_{20} - \pi, \qquad (37)$$

where Θ_{ij} is the dihedral angle between planes i and j. If Φ_{ij} is the angle between the normals to planes i and j, $\Theta_{ij} = \pi - \Phi_{ij}$, so

$$\cos\Phi_{ij} = -\cos\Theta_{ij} = \sin\left(\Theta_{ij} - \frac{\pi}{2}\right) = \mathbf{n}_i \cdot \mathbf{n}_j = q. \qquad (38)$$

Then $\Theta_{ij} = (\pi/2) + \sin^{-1}q$, $\Omega = (\pi/2) + 3\sin^{-1}q$, and

$$F_3(0,0,0) = \frac{1}{8} + \frac{3}{4\pi}\sin^{-1}q. \qquad (39)$$

Equations (26), (32), and (39) then show that $I_T=0$, $p=F_1(0)=\frac{1}{2}$ is indeed a fixed point of Eq. (21), so that the renormalization group procedure gives the exact percolation threshold $p_c=1/2$.

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